Subquadratic-Space Query-Efficient Data Structures for Realizable Order Types

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November 6, 2018

Abstract

Realizable order types and abstract order types are combinatorial analogues of line arrangements and pseudoline arrangements. They only store information about the relative orientation of triples of lines and pseudolines. We give an optimal encoding for abstract order types that allows efficient query of the orientation of any triple: the encoding uses $O(n^2)$ bits and an orientation query takes $O(\log n)$ time in the word-RAM model. We show how to shorten the encoding to $o(n^2)$ bits for realizable order types. We show how to attain $o(\log n)$ query time at the expense of an extra $O(\log^\varepsilon n)$ factor in the size of the encoding. For realizable order types, the encoding remains subquadratic in size.

1 Introduction

At SoCG'86, Chazelle asked [22]:

> "How many bits does it take to know an order type?".

This question is of importance in Computational Geometry for the following two reasons. First, in many algorithms dealing with sets of points in the plane, the only relevant information about the points is the orientation (clockwise or counterclockwise) of the triples of points in the set [13]. Second, computers as we know them can only handle numbers with finite description and we cannot assume that they are going to handle arbitrary real numbers without some sort of encoding. The study of *robust* algorithms is focused on ensuring the correct solution of problems on finite precision machines (see the Chapter on this issue in The Handbook [30]).

The orientation of an input triple $((x_1, y_1), (x_2, y_2), (x_3, y_3))$ corresponds to the sign of the determinant

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}.$$

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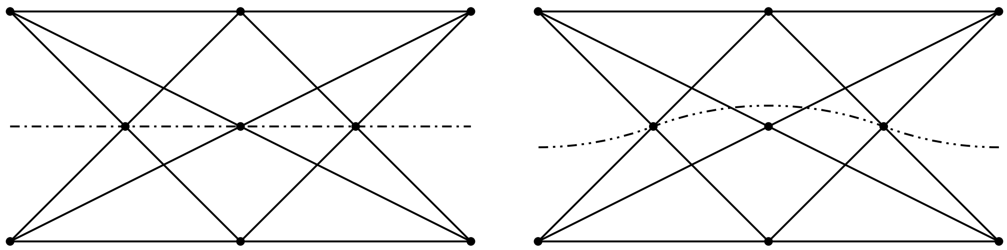

(a) Realizable order type.   (b) Abstract order type.

Figure 1: Pappus's configuration

The set of all orientations is usually referred to as the *order type* of the point set. A great deal of the literature in computational geometry deals with this notion [1–12, 14–23, 25–29], and the list goes on. The order type of a point set has been further abstracted into combinatorial objects known as (rank-three) *oriented matroids* [16]. The *chirotope axioms* define consistent systems of signs of triples [10]. From the Topological Representation Theorem [11], all such *abstract* order types correspond to pseudoline arrangements, while, from the standard projective duality, order types of point sets correspond to straight line arrangements.

In this contribution, we are interested in *compact* data structures for order types. We consider both order types that are realizable as point sets, and abstract order types, that is, order types realizable as pseudoline arrangements. We wish to design data structures using as few bits as possible that can be used to quickly answer orientation queries.

Abstract order types are much more numerous than realizable order types: there are $2^{\Theta(n^2)}$ abstract order types [14] and only $2^{\Theta(n \log n)}$ realizable order types [9, 20]. Hence information theory tells us that we need quadratic space for abstract order types whereas we only need linearithmic space for realizable order types. This discrepancy stems from the algebraic nature of realizable order types. As an example, Pappus gives a configuration where eight triples of concurrent straight lines force a ninth, whereas the ninth triple cannot be enforced by pseudolines [26, 29] (see Figure 1).

An obvious idea for storing an order type of a set of points (aside from explicitly storing all $\binom{n}{3}$ orientations) is to store the coordinates of the points, and answer orientation queries in constant time by computing the determinant. While this should work in many practical settings, it cannot work for all point sets. Perles's configuration shows that some arrangement of points, containing collinear triples, forces at least one coordinate to be irrational [24] (see Figure 2). Even for points in general position, it is well known that some arrangements require doubly exponential coordinates, hence coordinates with exponential bitsizes [23].

Goodman and Pollack defined λ-matrices that can be used for encoding realizable order types using $O(n^2 \log n)$ bits [18]. They asked if the space requirements can be moved closer to the information theoretic lower bounds. Felsner and Valtr [14, 15] show how to encode abstract order types optimally via the wiring diagram of their corresponding allowable sequence (as defined in [17]). However, it is not known how to decode the orientation of one triple from any of those encodings in, say, sublinear time. Moreover, since the information theoretic lower bound for realizable order types is only $\Omega(n \log n)$, we must ask if this space bound is approachable for those order types if we wish simultaneously to keep orientations queries reasonably efficient.

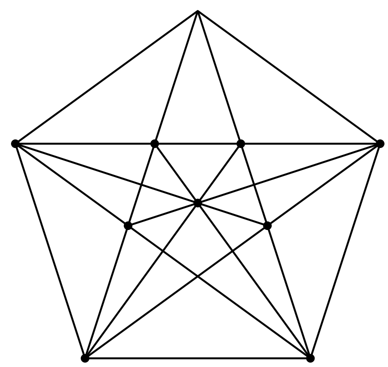

Figure 2: Perles's configuration

2 Our Results

We give the first optimal encoding for abstract order types that allows efficient query of the orientation of any triple: the encoding uses $O(n^2)$ bits and a query takes $O(\log n)$ time in the word-RAM model. Our data structure is far from being space-optimal for realizable order types and we can hope to save some space while keeping the queries fast. We show how to reduce the size of our data structure to subquadratic for realizable order types. We further refine our data structure so as to reduce the query time to sublogarithmic. This improvement is applicable for both abstract and realizable order types at the cost of an extra $O(\log^{\varepsilon} n)$ factor in the space complexity. For realizable order types, the encoding remains subquadratic in size.

Our data structure is the first subquadratic encoding of realizable order types that allows efficient query of the orientation of any triple. We know of no subquadratic constant-degree algebraic decision tree for the related problem of deciding whether a point set contains a collinear triple. Any such decision tree would yield another subquadratic encoding for realizable order types. We see our results as a stepping stone towards subquadratic nonuniform algorithms for this related problem.

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