Fourier Sampling

$H^{\otimes n} \longleftrightarrow$ Fourier transform on $\mathbb{Z}_2^n = \{0,1\}^n$

$x \in \{0,1\}^n \qquad x = (x_0, x_1, \dots, x_{n-1})$

$$x = x_0 + 2x_1 + 4x_2 + \dots + 2^{n-1} x_{n-1} \in [0, 2^n - 1]$$

$$|x\rangle = |x_0\rangle|x_1\rangle \dots |x_{n-1}\rangle = \bigotimes_{i=0}^{n-1} |x_i\rangle$$

$$H^{\otimes n}|x\rangle = \bigotimes_{i=0}^{n-1} H|x_i\rangle = \bigotimes_{i=0}^{n-1}\left[\frac{1}{\sqrt{2}} \sum_{y_i=0}^{1} (-1)^{x_i y_i} |y_i\rangle\right]$$

$H|x_0\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + (-1)^{x_0}|1\rangle\right]$

$= \frac{1}{\sqrt{2}} \sum_{y_0=0}^{1} (-1)^{x_0 y_0} |y_0\rangle \quad y \in \{0,1\}^n$

$$= \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} (-1)^{xy} |y\rangle \quad \text{where } x \cdot y = \sum_i x_i y_i$$

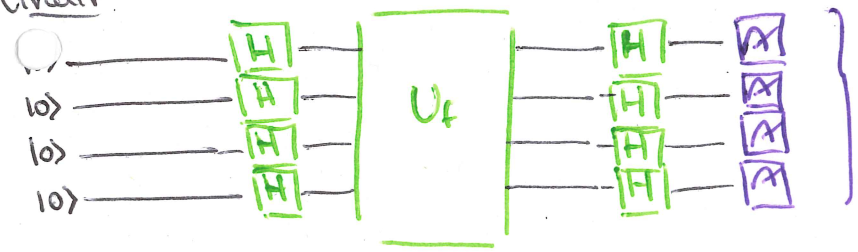
dot product

Suppose that $|\psi\rangle = \sum_{x=0}^{2^n-1} \psi(x)|x\rangle$

then $$H^{\otimes n}|\psi\rangle = \sum_{x=0}^{2^n-1} \psi(x) H|x\rangle = \sum_{x=0}^{2^n-1} \psi(x) \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} (-1)^{xy} |y\rangle$$

$$= \sum_{y=0}^{2^n-1} \overbrace{\left[\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} \psi(x)(-1)^{xy}\right]}^{\hat{\psi}(y)} |y\rangle = \sum_{y=0}^{2^n-1} \hat{\psi}(y)|y\rangle$$

Circuit



obtain y with prob. $|\hat{\Phi}(y)|^2$

$$|0\rangle^{\otimes n} \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle \xrightarrow{U_f} \underbrace{\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)}}_{= \Phi(x)} |x\rangle \xrightarrow{H^{\otimes n}} \sum_{y=0}^{2^n-1} \hat{\Phi}(y)|y\rangle$$

$$\Phi(x) = \frac{1}{\sqrt{2^n}} (-1)^{f(x)}$$

$$\hat{\Phi}(y) = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} \Phi(x)(-1)^{xy} = \frac{1}{2^n} \sum_{x=0}^{2^n-1} (-1)^{f(x)+xy}$$